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Convergent sequences of Legendre Padé approximants to the real and imaginary part of the scattering amplitudes

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Abstract. We extend the results of earlier work, by showing that convergent sequences of Legendre Padé approximants may be constructed for the real part of the scattering amplitude corresponding to a wide class of potentials including the pure Yukawa potential. Corresponding sequences of approximants to the imaginary part of the scattering amplitude are shown to converge for a wider class of potentials than proved previously.

1. Introduction

In previous papers (Common and Stacey 1978a, b) we investigated properties of approximants to Legendre series analogous to corresponding Padé approximants to power series. Applications were made in that work to summing the partial wave series of the scattering amplitude in potential scattering.

We were able to prove that sequences of our approximants to the imaginary part of the scattering amplitude converged for a certain class of potentials and also for the Coulomb scattering amplitude. It is the first purpose of this paper to extend these results by constructing convergent sequences of approximants to the real part of the amplitude. Our method is to start from the Mandelstam representation for the real part of the scattering amplitude

$$\begin{aligned}\operatorname{Re} F(s,t) = F_{\mathrm{B}}(t) &+ \sum_{i=0}^{M} \frac{\Gamma_i(t)}{(s+s_i)} \\ &+ \sum_{j=0}^{L_0-1} (t^j/\pi) \, P\!\!\int_0^\infty \rho_j(s')\,\mathrm{d}s'/(s'-s) \\ &+ (t^{L_0}/\pi^2) \, P\!\!\int_0^\infty \int_0^\infty \mathrm{d}s'\,\mathrm{d}t' \rho(s',t')/t'^{L_0}(s'-s)(t'-t) \end{aligned} \tag{1.1}$$

where $t = -2s(1-\cos\theta)$, $s =$ energy and θ is the scattering angle. M is the number of bound states with energies $s = -s_i$ and $\Gamma_i(t)$ are the residues at these poles. Finally $\rho(s, t)$ is the double spectral function and $\rho_j(s)$ the single spectral functions. Much work has been done in investigating when this representation can be proved to exist; in particular we used in our earlier work the result that this can be done (Bessis 1965) when the potential $V(r)$ is holomorphic in $\operatorname{Re} r > 0$ and bounded by

$$\begin{aligned} &|V(r)| < K|r|^{-\rho} && \text{for } |r| \leq 1, \rho < 2 \\ &|V(r)| < K|r|^{-\gamma} \exp(-\mu \operatorname{Re} r) && \text{for } |r| > 1, \mu > 0 \end{aligned} \tag{1.2}$$

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where $\gamma > \frac{7}{4}$. These results have been improved by Brander (1969a) to include those cases where $\frac{7}{4} \geq \gamma > \frac{5}{4}$.

However, as has been pointed out by Frederiksen *et al* (1975), in order for the principal value integrals to exist $\rho(s, t)$ must satisfy certain continuity conditions in s. Although in the above work it was stated that $\rho(s, t)$ is continuous in s, no precise continuity condition was given. As this continuity plays an important role in the proof of convergence of approximants, we start in § 2 by proving continuity in energy of the partial wave amplitudes and from this derive a corresponding property for $\rho(s, t)$. With this latter property we are then able to show that sequences of Legendre Padé approximants may be defined which converge to Re $F(s, t)$ for all potentials belonging to the class given by (1.2) but with $\gamma > \frac{5}{4}$. This result with the definition of the approximants are contained in the statement of theorem 2.5.

In § 3 we consider the class of potentials defined by

$$V(r) = (1/r) \int_{\mu}^{\infty} e^{-ur} \phi(u)\, du \tag{1.3}$$

with

$$\phi(u) \simeq O[(u-\mu)^{\gamma-2}]$$

where $1 < \gamma \leq \frac{5}{4}$ as $u \to \mu$ and $O(u^{\rho-2})$ with $\rho < 2$ as $u \to \infty$. These potentials satisfy equation (1.2) and have the holomorphy property. We show that theorem (2.5) holds for this class of potentials as well and also the important case of the pure Yukawa potential

$$V(r) = G \exp(-\mu r)/r.$$

It has been shown by Brander (1969b) that, although $\rho(s, t)$ is not a continuous function of t for potentials given by equation (1.3), its singularities are integrable for $\frac{3}{4} < \gamma \leq \frac{5}{4}$. We will give in § 3 a different method for obtaining this result and show that our approximants to Im $F(s, t)$ defined previously (Common and Stacey 1978b) converge for this extended class of potentials.

In § 4 we describe briefly the conclusions of this work, and finally give in several appendices details of some of the mathematical proofs.

2. Approximants to Re $F(s, t)$

We start by defining $f(\nu, k)$ to be an analytic continuation of the partial wave $f_l(s)$ into the complex $v = l + \frac{1}{2}$ plane with $s = k^2$. The following two results hold for the class of potentials given by equation (1.2):

Theorem 2.1. Let C be the contour in figure 1 such that all poles of $f(\nu, k)$ are to the left of it for all $k > 0$, and having right-hand extremity $L_0 - \sigma$ with L_0 a positive integer and $0 < \sigma < \frac{1}{2}$. If $V(r)$ satisfies the condition (1.2) with $\gamma > \frac{1}{2}$, then for all $k > 0$ and ν on C,

$$|f(\nu, k)| < \lambda_1 |\nu|^{\gamma - \frac{1}{2}} \tag{2.1}$$

where λ_1 is a constant independent of ν and k.

Proof: This bound follows immediately from the inequalities (4.13) of Brander 1969a).

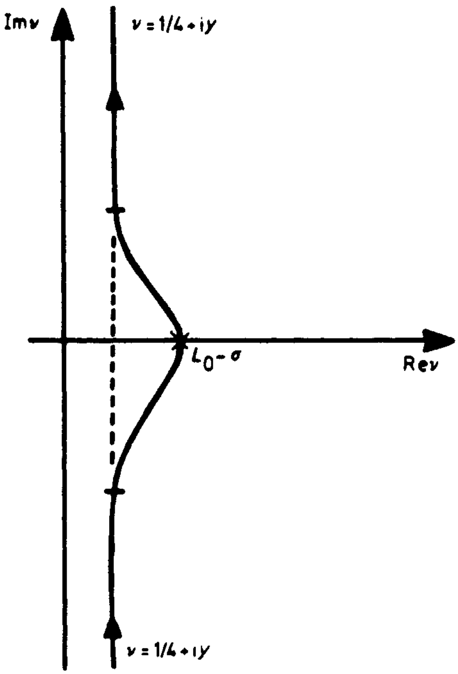

Figure 1. The integration contour C.

Theorem 2.2. If $V(r)$ satisfies the conditions (1.2) with $\gamma \geqslant 1$, then for all ν on C

$$|f(\nu, k+\delta) - f(\nu, k)| \leqslant \lambda_2(k)\delta/|\nu|^{\gamma-1} \tag{2.2}$$

where $\delta > 0$ and λ_2 is independent of ν but, as indicated, can depend on k.

The proof, which is rather long, will be given in Appendix A. It is based on the methods of Bessis (1964, 1965) used to obtain bounds on $|f(\nu, k)|$.

Let us now consider the double spectral function $\rho(s, t)$ which has for $\gamma > \frac{5}{4}$ from equation (4.10) of Brander (1969a) the representation

$$\rho(s, t) = (-\mathrm{i}/k) \int_{\mathrm{C}} f(\nu, k) f^*(\nu^*, k) P_{\nu-\frac{1}{2}}(1 + t/2k^2)\nu \, \mathrm{d}\nu. \tag{2.3}$$

From this representation Brander derived the bound†

$$|\rho(s, t)| < \lambda_3 s^{L_0-\sigma} t^{-1/4} (1+s)^{-2L_0+2\sigma-\frac{1}{4}} (s+t)^{L_0-\frac{1}{4}-\sigma} \tag{2.4}$$

valid for all $s > 0$, $t \geqslant 4\mu^2 + \mu^4/s$ when the potential is non-singular at $r = 0$.

Using the bounds given by theorems 2.1 and 2.2 we prove the corresponding continuity condition on the double spectral function.

Theorem 2.3. If $V(r)$ satisfies the conditions (1.2) with $\gamma > \frac{5}{4}$, then for given $\delta > 0$ and all $t \geqslant 4\mu^2 + \mu^4/s$ there exists $\epsilon_2 > 0$ such that

$$|\rho((k+\delta)^2, t) - \rho(k^2, t)| < \lambda_4(k)\delta^{\epsilon_2} t^{L_0-\sigma-\frac{1}{4}} \tag{2.5}$$

where λ_4 is independent of t and δ.

† Since Re $\nu < L_0 - \sigma$ on C, we change L_0 to $L_0 - \sigma$ in Brander's bound.

The proof again is rather long and will be given in Appendix B.

Using the above bounds we can follow the well worn path of deriving the Mandelstam representation (1.1). From the bound (2.4) it follows that $\operatorname{Im} F(s, t)$ satisfies the dispersion relation

$$\operatorname{Im} F(s, t) = \sum_{j=0}^{L_0 - 1} \rho_j(s) + (t^{L_0}/\pi) \int_{4\mu^2 + \mu^4/s}^{\infty} \rho(s, t')\, \mathrm{d}t'/t'^{L_0}(t' - t). \tag{2.6}$$

Then equation (1.1) follows if the double integral exists. We will show this in the following theorem which uses the bounds (2.5) and (2.4).

Theorem 2.4. If $V(r)$ satisfies the conditions (1.2) with $\gamma > \frac{5}{4}$, then the double integral in equation (1.1) exists for $t < 4\mu^2$ and $s > 0$, and

$$P\!\!\int_0^{\infty} [t^{L_0}/(s' - s)] \left[\int_{4\mu^2 + \mu^4/s'}^{\infty} \rho(s', t')\, \mathrm{d}t'/t'^{L_0}(t' - t) \right] \mathrm{d}s' = t^{L_0} \int_{4\mu^2}^{\infty} \psi(s, t')\, \mathrm{d}t'/(t' - t)t'^{L_0} \tag{2.7}$$

where

$$\psi(s, t) = P\!\!\int_0^{\infty} \rho(s', t)\, \mathrm{d}s'/(s' - s). \tag{2.8}$$

For all $t \geq 4\mu^2$,

$$|\psi(s, t)| \leq \lambda_5(s) t^{L_0 - \sigma - \frac{1}{2}} \tag{2.9}$$

with λ_5 independent of t.

Proof: Let I be the integral on the LHS of (2.7) and I_1, I_2 and I_3 the corresponding integrals over intervals $[0, s_1]$, $[s_1, s_2]$ and $[s_2, \infty]$ respectively where $0 < s_1 < s < s_2 < \infty$. Then

$$\begin{aligned} I_1 &= t^{L_0} \int_0^{s_1} [1/(s' - s)] \left[\int_{4\mu^2 + \mu^4/s'}^{\infty} \rho(s', t')\, \mathrm{d}t/t'^{L_0}(t' - t) \right] \mathrm{d}s' \\ &= t^{L_0} \int_{4\mu^2}^{\infty} [1/t'^{L_0}(t' - t)] \left[\int_0^{s_1} \rho(s', t')\, \mathrm{d}s'/(s' - s) \right] \mathrm{d}t' \end{aligned} \tag{2.10}$$

since the double integral is absolutely convergent as follows from (2.4). Also from this bound, it follows that

$$I_1 = t^{L_0} \int_{4\mu^2}^{\infty} [\psi_1(s, t')/t'^{L_0}(t' - t)]\, \mathrm{d}t' \tag{2.11}$$

with

$$|\psi_1(s, t')| = \left| \int_0^{s_1} \rho(s', t')\, \mathrm{d}s'/(s' - s) \right| \leq \lambda_6(s) t'^{L_0 - \sigma - \frac{1}{2}}. \tag{2.12}$$

Again using equation (2.4),

$$\begin{aligned} I_3 &= t^{L_0} \int_{s_2}^{\infty} [1/(s' - s)] \left[\int_{4\mu^2 + \mu^4/s'}^{\infty} \rho(s', t')\, \mathrm{d}t'/t'^{L_0}(t' - t) \right] \mathrm{d}s' \\ &= t^{L_0} \int_{4\mu^2}^{\infty} \psi_3(s, t')\, \mathrm{d}t'/t'^{L_0}(t' - t) \end{aligned} \tag{2.13}$$

with

$$|\psi_3(s,t')| \leqslant \left| \int_{s_2}^{\infty} \rho(s',t')\,\mathrm{d}s'/(s'-s) \right| \leqslant \lambda_7(s) t'^{L_0-\sigma-\frac{1}{2}}. \tag{2.14}$$

Finally

$$I_2 = t^{L_0} \mathrm{P}\!\!\int_{s_1}^{s_2} [1/(s'-s)] \left[\int_{4\mu^2+\mu^4/s'}^{\infty} \mathrm{d}t' \rho(s',t)/t'^{L_0}(t'-t) \right] \mathrm{d}s'$$

$$= t^{L_0} \ln((s_2-s)/(s-s_1)) \int_{4\mu^2+\mu^4/s}^{\infty} \mathrm{d}t' \rho(s,t')/t'^{L_0}(t'-t)$$

$$+ t^{L_0} \int_{s_1}^{s_2} [1/(s'-s)] \left\{ \int_{4\mu^2+\mu^4/s'}^{\infty} [\mathrm{d}t'(\rho(s',t')-\rho(s,t'))]/t'^{L_0}(t'-t) \right\}$$

since from equations (2.4) and (2.5) both single and double integrals are absolutely convergent. The order of integration in the repeated integral may therefore be interchanged so that

$$I_2 = t^{L_0} \int_{4\mu^2}^{\infty} [\mathrm{d}t'/t'^{L_0}(t'-t)] \left\{ \rho(s,t') \ln[(s_2-s)/(s-s_1)] \right.$$

$$\left. + \int_{s_1}^{s_2} [(\rho(s',t')-\rho(s,t'))\,\mathrm{d}s']/(s'-s) \right\}$$

$$= t^{L_0} \int_{4\mu^2}^{\infty} \psi_2(s,t')\,\mathrm{d}t'/t'^{L_0}(t'-t) \tag{2.15}$$

where

$$\psi_2(s,t') = \mathrm{P}\!\!\int_{s_1}^{s_2} [\rho(s',t)/(s'-s)]\,\mathrm{d}s' + \rho(s,t') \ln[(s_2-s)/(s-s_1)]. \tag{2.16}$$

Also using the bounds (2.4) and (2.5),

$$|\psi_2(s,t')| < \lambda_9(s) t'^{L_0-\sigma-\frac{1}{2}}. \tag{2.17}$$

Combining these representations and bounds for the I_i, equation (2.7) follows with $\psi(s,t)$ satisfying the bound (2.9).

Proceeding in the standard way, one substitutes equation (2.6) in the Khuri dispersion relation for $F(s,t)$ to obtain the Mandelstam representation (1.1) for $F(s,t)$. Theorem (2.4) shows that the double integral exists and that we may write equation (1.1) in the form

$$\mathrm{Re}\,F(s,t) = F_{\mathrm{B}}(t) + \sum_{i=0}^{\infty} \Gamma_i(t)/(s+s_i) + \sum_{j=0}^{L_0-1} (t^j/\pi)\, \mathrm{P}\!\!\int_0^{\infty} \rho_j(s')\,\mathrm{d}s'/(s'-s)$$

$$+ (t^{L_0}/\pi^2) \int_{4\mu^2}^{\infty} \psi(s,t')\,\mathrm{d}t'/t'^{L_0}(t'-t) \tag{2.18}$$

where $s>0$, $t \leqslant 4\mu^2$ and with $\psi(s,t)$ defined in equation (2.8).

It may be proved using unitarity (Blanckenbecker *et al* 1960) that the spin of the bound states cannot exceed L_0-1. Therefore the bound state terms in equation (2.18) only contribute to the partial waves up to $l = L_0 - 1$, and we may make the expansion

$$\mathrm{Re}F(s, t) = F_{\mathrm{B}}(t) + (1/k) \sum_{l=0}^{\infty} (2l+1)\ \mathrm{Re}(f_l(s) - f_l^{\mathrm{B}}(s))P_l(\cos\theta) \tag{2.19}$$

where $f_l^{\mathrm{B}}(s)$ are the partial waves of the Born amplitude. For $l \geqslant L_0$

$$\mathrm{Re}\, f_l(s) - f_l^{\mathrm{B}}(s) = \int_{x_0}^{\infty} Q_l(x)\Psi(x, s)\,\mathrm{d}x \qquad s > 0 \tag{2.20}$$

with

$$x_0 = 1 + 2\mu^2/k^2, \qquad \Psi(x, s) = (k/\pi^2)\psi[s, 2k^2(x-1)]. \tag{2.21}$$

From equation (2.9)

$$|\Psi(x, s)| \leqslant \lambda_{10}(s)x^{L_0 - \sigma - \frac{1}{2}} \text{ for } s > 0,\ x > x_0 \tag{2.22}$$

where $\lambda_{10}(s)$ is independent of x but may depend on s.

This weight function $\Psi(x, s)$ satisfies condition (5.12) of our previous work (Common and Stacey 1978b) and so we can construct a convergent set of Legendre Padé approximants to Re $F(s, t)$ in the same manner as for Im $F(s, t)$. The following theorem corresponds to theorem 5.1 of Common and Stacey (1978b) and its proof is exactly the same, so we will not give the details here.

Theorem 2.5. Let

$$\phi_3(v) = v^{L_0} \int_{x_0}^{\frac{1}{2}(v+1/v)} \Psi(x, s)\,\mathrm{d}x/(1 - 2vx + v^2)^{1/2} + (\mathrm{i}c/2)(rv - r^2v^2)^{-1/2} \tag{2.23}$$

where c is a real nonzero constant and $r_0 = x_0 + (x_0^2 - 1)^{1/2}$, and let

$$\begin{aligned} g(w) &\equiv \sum_{l=0}^{\infty} (\mathrm{Re}\, f_{l+L_0}(s) - f_{l+L_0}^{\mathrm{B}}(s))(-w)^l \\ &\equiv \int_0^{1/r_0} \phi_3(v)\,\mathrm{d}v/(1+vw) - (\mathrm{i}c/2)\int_0^{1/r_0} \mathrm{d}v/(r_0v - r_0^2v^2)^{1/2}(1+vw) \\ &\equiv N(w) - CS(w). \end{aligned} \tag{2.24}$$

If

$$g_n(w) = \sum_{p=1}^{2n} \alpha_p/(1 + \sigma_p w)$$

is the approximant to $g(w)$ formed by taking $(n - 1/n)$ Padé approximants to $N(w)$ and $S(w)$, then the sequence of approximants

$$\begin{aligned} \mathrm{Re}\, F_n^L(s, t) &= (1/k) \sum_{p=1}^{2n} \alpha_p'(1 - \sigma_p^2)/(1 - 2\sigma_p z + \sigma_p^2)^{3/2} \\ &+ (1/k)\sum_{l=0}^{L_0-1} (2l+1)\Big(\mathrm{Re}\, f_l(s) - f_l^{\mathrm{B}}(s) - \sum_{p=1}^{2n} \alpha_p'\sigma_p^l\Big)P_l(z) + F_{\mathrm{B}}(t) \qquad n = 0, 1, 2, \ldots \end{aligned}$$

where $\alpha_p' = \alpha_p\sigma_p^{-L_0}$ and $z = \cos\theta$, converge uniformly to Re $F(s, t)$ as $n \to \infty$ for fixed $s > 0$, for all t in any closed bounded domain of the complex t plane cut from $4\mu^2$ to ∞, when $\Psi(x, s)$ satisfies equation (2.22).

This theorem gives us a convergent set of approximants to the real part of the scattering amplitude for the class of potentials defined by (1.2) where $\gamma > \frac{5}{4}$.

3. Convergent sequences for extended classes of potentials

We will now sketch how one proceeds for $1 < \gamma \leq \frac{5}{4}$ and consider in some detail the important case of the Yukawa potential when $\gamma = 1$. The problem for these values of γ is that the path integral expression (2.3) for $\rho(s, t)$ is no longer absolutely convergent. The remedy is as suggested by Bessis (1965) to subtract off the contribution of the Born terms and consider

$$\rho_{\mathrm{R}}(s, t) \equiv (-\mathrm{i}/k) \int_{\mathrm{C}} (f(\nu, k) f^*(\nu^*, k) - f^{\mathrm{B}}(\nu, k) f^{\mathrm{B}^*}(\nu^*, k)) P_{\nu-\frac{1}{2}}(1 + t/2s) \nu \,\mathrm{d}\nu$$

$$= -\mathrm{i}/k \int_{\mathrm{C}} [(f(\nu, k) - f^{\mathrm{B}}(\nu, k)) f^*(\nu^*, k) + f^{\mathrm{B}}(\nu, k) \times (f^*(\nu^*, k) - f^{\mathrm{B}^*}(\nu^*, k))] P_{\nu-\frac{1}{2}}(1 + t/2s) \nu \,\mathrm{d}\nu. \tag{3.1}$$

The integral over C is convergent for $\gamma \geq 1$ since

$$|f(\nu, k) - f^{\mathrm{B}}(\nu, k)| \leq (\lambda_{11}/|\nu|) |(f(\nu, k)| \tag{3.2}$$

with λ_{11} independent of ν, k. Since $|f(\nu, k)|$ still satisfies the bound (2.1), it follows that $\rho_{\mathrm{R}}(s, t)$ also satisfies (2.4) and (2.5). The dispersion relation (2.6) for $\operatorname{Im} F(s, t)$ is replaced by

$$\operatorname{Im} F(s, t) = \sum_{j=0}^{L_0-1} \rho_{l,R}(s) + (t^{L_0}/\pi) \int_{4\mu^2+\mu^4/s}^{\infty} \rho_{\mathrm{R}}(s, t') \,\mathrm{d}t'/t'^{L_0}(t' - t) + (1/k) \sum_{l=0}^{\infty} (2l+1) |f_l^{\mathrm{B}}(k)|^2 P_l(1 + t/2s). \tag{3.3}$$

We now consider the class of potentials where

$$V(r) = (1/r) \int_{\mu}^{\infty} \exp(-ur) \phi(u) \,\mathrm{d}u. \tag{3.4}$$

From standard results on the limiting behaviour of Laplace transforms, it follows that $V(r)$ will satisfy the bounds (1.2) with $1 < \gamma \leq \frac{5}{4}$ and $1 < \rho < 2$ if as $u \to \mu$, $\phi(u) = \mathrm{O}[(u - \mu)^{\gamma-2}]$ and as $u \to \infty \phi(u) = \mathrm{O}(u^{\rho-2})$. Also $V(r)$ is holomorphic for $\operatorname{Re} r > \mu$. So we will consider this class of potentials† and the important pure Yukawa potential $V(r) = V_0[\exp(-\mu r)]/r$.

We show in Appendix C that

$$\sum_{l=0}^{\infty} (2l+1) |f_l^{\mathrm{B}}(k)|^2 P_l(1 + t/2s) = \sum_{l=0}^{\infty} (2l+1) |f_l^{\mathrm{B}}(k)|^2 + (t/\pi) \int_{4\mu^2+\mu^4/s}^{\infty} \rho_{\mathrm{B}}(s, t') \,\mathrm{d}t'/t'(t' - t) \tag{3.5}$$

† This class of potentials is more restricted than that given by (1.2) since the above relations between the limiting behaviour of $\phi(u)$ and $V(r)$ are not reversible.

where $\rho_{\mathrm{B}}(s, t')$ satisfies the conditions

$$
\begin{aligned}
|\rho_{\mathrm{B}}(s, t)| \leqslant \eta_9 \theta(t-4\mu^2-\mu^4/k^2)[&k^{\frac{3}{2}-2\gamma} t^{\gamma/2-\frac{3}{4}}(2+t/2k^2)^{2-\frac{3}{2}\gamma}(1+\mu^2/k^2)^{\frac{5}{2}-2\gamma} \\
&\times(t-4\mu^2-\mu^4/k^2)^{-\frac{5}{2}+2\gamma} \\
&+t^{\rho/2-\frac{1}{2}}/k+t^{\gamma-\frac{3}{4}}/k+k^{-1/2}(4k^2+\mu^2)^{-1/4} t^{\frac{1}{2}-\gamma/2}(2+t/2k^2)^{\frac{3}{4}-\gamma/2}]
\end{aligned} \tag{3.6}
$$

and

$$
\begin{aligned}
&|\rho_{\mathrm{B}}(s, t)-\rho_{\mathrm{B}}(s_1, t)| \\
&\quad \leqslant \eta_{10}(s, s_1)\{[|s-s_1|/(k^2+\mu^2)]|\rho_{\mathrm{B}}(s, t)|+(|s-s_1|^{\delta_1}/|k^2+\mu^2|^{1/2}) t^{2\delta_1} \\
&\quad \times[t^{\frac{5}{4}+3\delta_1/2-\gamma}(t-4\mu^2-\mu^4/k^4)^{2\gamma-\frac{5}{2}-\delta_1}+t^{\rho/2-\frac{1}{2}+\delta_1}+t^{\gamma-\frac{3}{4}-\delta_1}+t^{\frac{5}{4}-\gamma+3\delta_1/2}]\} \\
&\quad \times\theta(t-4\mu^2-\mu^4/k^2)+(s_1 \Leftrightarrow s)
\end{aligned} \tag{3.7}
$$

where η_{10} is independent of t and η_9 is independent of s, t. Also $0<\delta_1<\frac{1}{2}$.

Using these conditions and the corresponding conditions on $\rho_{\mathrm{R}}(s, t)$ the following theorem may be proved.

Theorem 2.4.1. If

$$
V(r)=r^{-1} \int_\mu^{\infty} \phi(u) \exp(-ur)\, \mathrm{d}u \tag{3.8}
$$

where

$$
\phi(u) \simeq \mathrm{O}[(u-\mu)^{\eta-2}]
$$

with $1<\gamma \leqslant \frac{5}{4}$ as $u \rightarrow \mu$ and $\mathrm{O}[u^{\rho-2}]$ with $1<\rho<2$ as $u \rightarrow \infty$, then for $t \notin (4\mu^2, \infty)$,

$$
\begin{aligned}
&\mathrm{P}\!\!\int_0^{\infty}[t^{L_0}/(s'-s)]\left[\int_{4\mu^2+\mu^4/s'}^{\infty}[\rho_{\mathrm{B}}(s', t') k'^{-1}+\rho_{\mathrm{R}}(s', t')]/t'^{L_0}(t'-t)\right] \mathrm{d}t'\, \mathrm{d}s' \\
&\quad =t^{L_0} \int_{4\mu^2}^{\infty}[\psi_1(s, t')\, \mathrm{d}t'/(t'-t) t'^{L_0}]
\end{aligned} \tag{3.9}
$$

where

$$
\psi_1(s, t')=\mathrm{P}\!\!\int_0^{\infty}\{[\rho_{\mathrm{B}}(s', t') k'^{-1}+\rho_{\mathrm{R}}(s', t')]/(s'-s)\}\, \mathrm{d}s' . \tag{3.10}
$$

All integrals exist and for all $t>4\mu^2$

$$
\begin{aligned}
|\psi_1(s, t)| \leqslant\, &\lambda_{12}(s) t^{L_0-\sigma-\frac{1}{2}} \\
&+\lambda_{13}(s) t^{\frac{5}{4}-\gamma+3\delta_1/2} \theta(t-4\mu^2-\mu^4/k^2)/(t-4\mu^2-\mu^4/k^2)^{\frac{5}{2}-2\gamma+\delta_1}
\end{aligned} \tag{3.11}
$$

where as indicated λ_{12}, λ_{13} are independent of t.

Let us now consider the pure Yukawa potential $V(r)=G[\exp(-\mu r)]/r$. Once again the $\operatorname{Im} F(s, t)$ has the representation given by (3.3), with $\rho_{\mathrm{R}}(s, t)$ satisfying the usual bounds.

In this case

$$\operatorname{Im} F_{2B}(s,t) \equiv (1/k) \sum_{l=0}^{\infty} (2l+1)|f_l^{B}(k)|^2 P_l(1+t/2s)$$

$$= (G^2/k^3) \sum_{l=0}^{\infty} (2l+1)[Q_l(1+\mu^2/2k^2)]^2 P_l(1+t/2s)$$

$$= 2G^2 \int_{4\mu^2+\mu^4/k^2}^{\infty} [1/(t'-t)(t')^{1/2}(t'-4\mu^2-\mu^4/k^2)^{1/2}]\,dt' \tag{3.12}$$

and (Goldberger and Watson 1964)

$$\operatorname{Re} F_{2B}(s,t) \equiv \pi^{-1} P\!\int_0^{\infty} \operatorname{Im} F_{2B}(s',t)/(s'-s)$$

$$= 2G^2 \int_{4\mu^2}^{4\mu^2+\mu^4/s} dt'[1/(t')^{1/2}(t'-t)][1/(4\mu^2+\mu^4/k^2-t')^{1/2}] \tag{3.13}$$

$$= \sum_{l=0}^{L_0} t^l b_l(s) + (t^{L_0+1}/\pi) \int_{4\mu^2}^{4\mu^2+\mu^4/s} \psi_2(s,t')\,dt'/t'^{L_0+1}(t'-t)$$

where in this case the weight function

$$\psi_2(s,t') \equiv 2\pi G^2 \theta(4\mu^2+\mu^4/k^2-t')/(t')^{1/2}(4\mu^2+\mu^4/k^2-t')^{1/2}. \tag{3.14}$$

Finally let us go back to the case of $\operatorname{Im} F(s,t)$ which we considered in our previous work (Common and Stacey 1978b) where we obtained convergent sequences of Legendre Padé approximants when $V(r)$ satisfied equation (1.2) with $\gamma > \frac{7}{4}$. We will extend this result to the cases $\frac{3}{4} < \gamma < \frac{7}{4}$. Here we use the representation (2.6) for $\operatorname{Im} F(s,t)$ i.e.

$$\operatorname{Im} F(s,t) = \sum_{j=0}^{L_0-1} \rho_j(s) + (t^{L_0}/\pi) \int_{4\mu^2+\mu^4/s}^{\infty} \rho(s,t')\,dt'/t'^{L_0}(t'-t)$$

with from (2.4)

$$|\rho(s,t)| \leqslant \lambda_{13}(s) t^{L_0-\sigma-\frac{1}{2}} \qquad \tfrac{7}{4} \geqslant \gamma > \tfrac{5}{4} \tag{3.15}$$

and from (3.6), (2.4)

$$|\rho_R(s,t)+\rho_B(s,t)| \leqslant \lambda_{14}(s) t^{L_0-\sigma-\frac{1}{2}}$$

$$+\lambda_{15}(s) t^{\frac{5}{4}-\gamma}/(t-4\mu^2-\mu^4/s)^{\frac{5}{2}-2\gamma} \qquad \tfrac{3}{4} < \gamma \leqslant \tfrac{5}{4} \tag{3.16}$$

when $t > 4\mu^2+\mu^4/s$, where the λ are independent of t.

Using these representations and bounds, we will construct convergent sequences of Padé approximants to the respective amplitudes. We start by considering $\operatorname{Im} F(s,t)$ above when $\frac{7}{4} \geqslant \gamma > \frac{3}{4}$.

Theorem 3.1. Let

$$\phi_4(t) = v^{L_0+1} \int_{x_0}^{\frac{1}{2}(v+1/v)} \Psi(x,s)\,dx/(1-2vx+v^2)^{1/2} + (ic/2)(r_0 v - r_0^2 v^2)^{-1/2} \tag{3.17}$$

where c is a real nonzero constant and let

$$g(w) \equiv \sum_{l=0}^{\infty} \operatorname{Im} f_{l+L_0+1}(s)(-w)^l$$
$$\equiv \int_0^{1/r_0} \phi_4(v)\,\mathrm{d}v/(1+vw) - (\mathrm{i}c/2)\int_0^{1/r_0} \mathrm{d}v/(r_0v - r_0^2v^2)^{1/2}(1+vw)$$
$$\equiv N(w) - cs(w) \tag{3.18}$$

where $r_0 = x_0 + (x_0^2 - 1)^{1/2}$ and $x_0 = 1 + (1/2k^2)(4\mu^2 + \mu^4/k^2)$.

The weight function is $\Psi(x, s) = (k/\pi^2)\rho[s, 2k^2(x-1)]$. If

$$g_n(s) = \sum_{p=1}^{2n} \sigma_p/(1+\sigma_p w)$$

is the approximant to $g(w)$ formed by taking $(n-1/n)$ Padé approximants to $N(w)$ and $S(w)$, then the sequence of approximants

$$\operatorname{Im} F_n^L(s,t) = (1/k)\sum_{p=1}^{2n} [\alpha'_p(1-\sigma_p^2)/(1-2\sigma_p z + \sigma_p^2)^{3/2}] + 1/k \sum_{l=0}^{L_0-1} (2l+1)\operatorname{Im} f_l(s) P_l(z) \tag{3.19}$$

where $\sigma'_p = \alpha_p\sigma_p^{-L_0}$ and $z = 1 + t/2k^2$, converge uniformly to $\operatorname{Im} F(s,t)$ as $n \to \infty$ for fixed $s > 0$, for all t in any closed bounded domain of the complex t plane cut from $4\mu^2$ to ∞, when $V(r)$ has the representation (3.8) with $\frac{3}{4} < \gamma \leq \frac{5}{4}$ or when $V(r)$ satisfies (1.2) with $\frac{5}{4} < \gamma \leq \frac{7}{4}$.

Proof: For $\frac{5}{4} < \gamma \leq \frac{7}{4}$, the proof follows precisely the proof of theorem 5.1 of our previous work (Common and Stacey 1978b), taking into account the bound (3.15).

For $\frac{3}{4} < \gamma \leq \frac{5}{4}$, the only problem is the singularity at $t = 4\mu^2 + \mu^4/s$. We may write

$$\Psi(x,s) = \Psi_1(x,s) + \Psi_2(x,s)$$

where

$$|\Psi_2(x,s)| \leq c_2(s)x^{L_0-\sigma-\frac{1}{2}} \qquad \sigma > 0 \tag{3.20}$$
$$|\Psi_1(x,s)| \leq c_1(s)(x-x_0)^{2\gamma-\frac{5}{2}}\theta(x_1 - x)$$

and $x_0 = 1 + 2\mu^2/s + \mu^4/2s^2$, where x_1 is fixed to be greater than x_0.

To prove the theorem we have to show that $\phi_4(v)$ satisfies the conditions that allow the applications of a convergence theorem for Padé approximants by Nuttall (1977) given in our previous work (Common and Stacey 1978b). This theorem has been published in a slightly modified form in a paper by Nuttall and Wherry (1978). The above conditions are that if $h(\theta) = |\sin\theta|\,|\phi_4[(1/2r_0)(1+\cos\theta)]|$, then for all $0 \leq \theta \leq \pi$

$$0 < M_2 < |h(\theta)| \leq M_1 \tag{3.21}$$

and

$$|h(\theta - \delta\theta) - h(\theta)| \leq M_3|\ln \delta\theta|^{-\alpha} \qquad \alpha > 1. \tag{3.22}$$

As in our previous work the contribution of $\Psi_2(x,s)$ to $\phi_4(v)$ and hence $h(\theta)$ will satisfy equation (3.22) and the RHS inequality of equation (3.21). The LHS inequality of equation (3.21) will be satisfied by the complete $h(\theta)$ as previously described.

Let $h_1(\theta)$ be the contribution to $h(\theta)$ corresponding to $\Psi_1(x, s)$. Then

$$h_1(\theta) = 2(r_0 v - v^2 r_0^2)^{1/2} v^{L_0+1} \int_{x_0}^{\frac{1}{2}(v+1/v)} \Psi_1(x, s)\, dx/(1 - 2vx + v^2)^{1/2} \tag{3.23}$$

where $v = (1/2r_0)(1 + \cos\theta)$. From equation (3.20)

$$|h_1(\theta)| \leqslant c_1(s)(r_0 v - r_0^2 v^2)^{1/2} v^{L_0+1} \int_{x_0}^{\frac{1}{2}(v+1/v)} (x - x_0)^{2\gamma - \frac{5}{2}} \theta(x_1 - x)\, dx/(1 - 2vx + v^2)^{1/2} \tag{3.24}$$

For $\gamma > \frac{3}{4}$, the integral is uniformly bounded for all $0 \leqslant v \leqslant 1/r_0$, so that $|h_1(\theta)|$ satisfies the RHS inequality of equation (3.21).

Let us now consider the continuity condition for $0 < \theta < \pi$ first of all†.

$$\begin{aligned} &|h_1(\theta) - h_1(\theta + \delta\theta)| \\ &\quad \leqslant 2|\{[r_0(v + \delta v) - (v + \delta v)^2 r_0^2]^{1/2}/(r_0 v - v^2 r_0^2)^{1/2}\}(v + \delta v)^{L_0+1} - 1|\, |h_1(\theta + \delta\theta)| \\ &\quad + |2(r_0 v - v^2 r_0^2)^{1/2} v^{L_0+1}| \left| \int_{x_0}^{\frac{1}{2}(v+1/v)} \Psi_1(x, s)\, dx/(1 - 2vx + v^2)^{1/2} \right. \\ &\quad \left. - \int_{x_0}^{\frac{1}{2}(v+\delta v)+[2(v+\delta v)]^{-1}} \Psi_1(x, s)\, dx/[1 - 2(v + \delta v)x + (v + \delta v)^2]^{1/2} \right| \end{aligned} \tag{3.25}$$

where

$$\delta v = (1/2r_0)[\cos(\theta + \delta\theta) - \cos\theta] \simeq (-\sin\theta/2r_0)\delta\theta.$$

The first term on the RHS of equation (3.25) is $O(\delta v)$ for $0 < v < 1/r_0$ since $|h_1(\theta + \delta\theta)|$ is bounded by a constant.

Also

$$\begin{aligned} &\left| \int_{x_0}^{\frac{1}{2}(v+1/v)} \Psi_1(x, s)\, dx/(1 - 2vx + v^2)^{1/2} \right. \\ &\quad \left. - \int_{x_0}^{\frac{1}{2}(v+\delta v)+[2(v+2v)]^{-1}} \Psi_1(x, s)\, dx/[1 - 2(v + \delta v)x + (v + \delta v)^2]^{1/2} \right| \\ &\leqslant c_1(s) \left| \int_{x_0}^{\frac{1}{2}(v+1/v)} \frac{\theta(x_1 - x)}{(x - x_0)^{(\frac{5}{2} - 2\gamma)}} \right| \\ &\quad \times \left| \frac{1}{(1 - 2vx + v^2)^{1/2}} - \frac{1}{[1 - 2(v + \delta v)x + (v + \delta v)^2]^{1/2}} \right| dx \\ &\quad + c_1(s) \int_{\frac{1}{2}(v+1/v)}^{\frac{1}{2}(v+\delta v)+[2(v+\delta v)]^{-1}} \frac{dx}{(x - x_0)^{(\frac{5}{2} - 2\gamma)}[1 - 2(v + \delta v)x + (v + \delta v)^2]^{1/2}}. \end{aligned} \tag{3.26}$$

It is straightforward to prove that both terms on the RHS of this inequality are $O[(\delta v)^{1/2}]$. Therefore for $0 < \theta < \pi$,

$$|h_1(\theta) - h_1(\theta + \delta\theta)| = O[(\delta v)^{1/2}]. \tag{3.27}$$

† It is assumed that $\delta\theta$ is chosen sufficiently small so that $0 < \theta + \delta\theta < \pi$.

It is easy to show that $h_1(\theta)\to 0$ as $\theta\to\pi$. Therefore when $\theta=\pi$,

$$|h_1(\theta)-h_1(\theta-\delta\theta)|=|h_1(\theta-\delta\theta)|$$
$$=(\delta v)^{L_0+\frac{3}{2}}\int_{x_0}^{x_1}\Psi_1(x)\,\mathrm{d}x/[1-2\delta vx+(\delta v)^2]^{1/2} \tag{3.28}$$
$$=\mathrm{O}[(\delta\theta)^{2L_0+3}].$$

Similarly when $\gamma>\frac{3}{4}$, $h_1(\theta)\to 0$ as $\theta\to 0$ i.e. $v\to 1/r_0$ so that

$$|h_1(\theta)-h_1(\theta+\delta\theta)|$$
$$\leqslant h_1(\theta+\delta\theta)\sim c_1(s)[r_0v-r_0^2(v+\delta v)^2]^{1/2}(v+\delta v)^{L_0+1}$$
$$\times\int_{x_0}^{x_1}\Psi_1(x,s)\,\mathrm{d}x/[1-2(v+\delta v)x+(v+\delta v)^2]^{1/2}$$
$$=\mathrm{O}[(\delta v)^{2\gamma-\frac{3}{2}}]=\mathrm{O}[(\delta\theta)^{4\gamma-3}]. \tag{3.29}$$

From the three inequalities (3.27), (3.28) and (3.29) we see that $h_1(\theta)$ satisfies (3.21) so long as $\gamma>\frac{3}{4}$. Hence $h(\theta)$ satisfies the conditions (3.21), (3.22) so that as in our previous work the sequence of approximants defined by (3.19) converge uniformly to Im $F(s,t)$ as $n\to\infty$ for fixed $s>0$, for all t in any closed bounded domain of the complex t plane cut from $4\mu^2$ to ∞.

Finally we give the corresponding result for the approximants to the real part of the scattering amplitude.

Theorem 3.2. Let Re $F(s,t)$ be the scattering amplitude corresponding to a potential $V(r)$ having the representation (3.8) with $1<\gamma\leqslant\frac{5}{4}$, $1<\rho<2$, or corresponding to the Yukawa potential $V(r)=G[\exp(-\mu r)/r]$. Then the approximants Re $F_n^L(s,t)$ defined by equation (2.25) converge uniformly to Re $F(s,t)$ as $n\to\infty$ for fixed s, for all t in any closed bounded domain of the complex t plane cut from $4\mu^2$ to ∞.

Proof: The result follows from the bounds (3.11) and (3.14) for the respective weight functions $\psi_1(s,t)$, $\psi_2(s,t)$. As in theorem 3.1 the singular parts of these bounds give no difficulties and the proof follows that of the above theorem.

4. Conclusions

We have extended in this work the class of potentials for which convergent sequences of Legendre Padé approximants to the imaginary part of the scattering amplitude may be defined. We have also shown that convergent sequences of these approximants to the real part of the scattering amplitude may be defined for a wide class of potentials, including the pure Yukawa potential. In a following paper numerical results will be presented for this very important case of the Yukawa potential.

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Appendix A

We prove here the continuity condition (2.2) on $f(\nu, k)$ for all points on C when $V(z)$ satisfies bounds (1.2) with $\gamma \geq 1$. The method follows exactly that used by Bessis (1964) in the proof of bounds on $|f(\nu, k)|$ so we will only sketch the proof of the condition (2.2), referring to the above work for the details.

From the holomorphy property of $V(r)$ in Re r and from the bounds (1.2), it follows that

$$\begin{aligned} |\mathrm{d}V/\mathrm{d}r| &\leq K_1/|r|^{\rho+1} \qquad 0<|r|\leq 1 \\ &\leq K_1[\exp(-\mu_1 \operatorname{Re} r)]/|r|^{\gamma+1} \qquad |r|\geq 1 \end{aligned} \tag{A.1}$$

where K_1, μ_1 are constants with $0<\mu_1<\mu$. Remember that $\rho<2$ and we are taking $\gamma \geq 1$.

We make the definition $V(k, z) \equiv (1/k^2) V(z/k)$ and we can deduce from equation (A.1) that

$$\begin{aligned} &|V(k, z) - V(k+\delta, z)| \\ &\quad \leq (K_2 \delta k^{\rho-3}/|z|^{\rho})(1+1/k^2) \qquad |z|\leq k \\ &\quad \leq [K_2 \delta k^{\gamma-3}/|z|^{\gamma}](1+k^2)^{-\mu_1 \operatorname{Re} z/k} \qquad |z|>k \end{aligned} \tag{A.2}$$

where K_2 is a constant.

To deduce the corresponding continuity condition on $f(\nu, k)$ we start from the Volterra equation for the regular solution $u_{\nu-\frac{1}{2}}(k, z)$ to the Schrödinger radial equation. This is (Bessis 1964)

$$\begin{aligned} u_{\nu-\frac{1}{2}}(k, z) = (\pi z/2)^{1/2} J_\nu(z) + \mathrm{i} \int_0^z & [(\pi z/2)^{1/2} J_\nu(z) (\pi z'/2)^{1/2} H_\nu^{(1)}(z') \\ & - (\pi z'/2)^{1/2} J_\nu(z')(\pi z/2)^{1/2} H_\nu^{(1)}(z)] V(k, z') u_{\nu-\frac{1}{2}}(k, z')\, \mathrm{d}z'. \end{aligned} \tag{A.3}$$

The amplitude $f(\nu, k)$ is given by

$$f(\nu, k) = -A(\nu, k)/[1+\mathrm{i}B(\nu, k)] \tag{A.4}$$

where

$$\begin{aligned} A(\nu, k) &= \int_0^\infty (\pi z/2)^{1/2} J_\nu(z) V(k, z) u_{\nu-\frac{1}{2}}(k, z)\, \mathrm{d}z \\ B(\nu, k) &= \int_0^\infty (\pi z/2)^{1/2} H_\nu^{(1)}(z) V(k, z) u_{\nu-\frac{1}{2}}(k, z)\, \mathrm{d}z. \end{aligned} \tag{A.5}$$

To prove continuity of $f(\nu, k)$, we have therefore to prove continuity of $u_{\nu-\frac{1}{2}}(k, z)$.

From equation (A.3)

$$u_{\nu-\frac{1}{2}}(k+\delta, z) - u_{\nu-\frac{1}{2}}(k, z)$$
$$= \mathrm{i}\int_0^z [J, H][V(k+\delta, z') - V(k, z')]u_{\nu-\frac{1}{2}}(k, z')\,\mathrm{d}z'$$
$$+\mathrm{i}\int_0^z [J, H]V(k+\delta, z')[u_{\nu-\frac{1}{2}}(k+\delta, z') - u_{\nu-\frac{1}{2}}(k, z')]\,\mathrm{d}z' \quad \text{(A.6)}$$

where

$$[J, H] = (\pi z/2)^{1/2}J_\nu(z)(\pi z'/2)^{1/2}H_\nu^{(1)}(z') - (\pi z'/2)^{1/2}J_\nu(z')(\pi z/2)^{1/2}H_\nu^{(1)}(2).$$

The integrals from 0 to ∞ in equation (A.5) are not taken along the real axis, but along a contour $\Gamma(\nu)$ which runs along the straight line from 0 to ν and then along the anti-Stokes line S for Bessel functions from ν to ∞ (Bessis 1964). We can arrange the contour C so that on it $|\nu| \geqslant 1$. Then on $\Gamma(\nu)$ one has the bounds

$$\left.\begin{array}{l} |J_\nu(z)| < \{c_1 \exp[-\mathrm{Re}\,\nu(\alpha - \tanh\alpha)]/|\nu^2 - z^2|^{1/4}\}(2/\pi)^{1/2} \\ |H_\nu^{(1)}(z)| < \{c_1 \exp[+\mathrm{Re}\,\nu(\alpha - \tanh\alpha)]/|\nu^2 - z^2|^{1/4}\}(2/\pi)^{1/2} \end{array}\right\} \quad \text{for } z = \nu/\cosh\alpha,\ \alpha \geqslant 0$$
$$\left.\begin{array}{l} |J_\nu(z)| < [c_1/|\nu^2 - z^2|^{1/4}](2/\pi)^{1/2} \\ |H_\nu^{(1)}(z)| < (c_1/|\nu^2 - z^2|^{1/4})(2/\pi)^{1/2} \end{array}\right\} \quad \text{for } z \in S. \quad \text{(A.7)}$$

with $c < 23$, a purely numerical constant.

Using these bounds in (A.3), Bessis obtained the bounds

$$|u_{\nu-\frac{1}{2}}(k, z)| \leqslant c_1 \exp[-\mathrm{Re}\,\nu(\alpha - \tanh\alpha)]|z|^{1/2}/|\nu^2 - z^2|^{1/4}$$
$$\times \exp\left[2c_1^2 \int_0^z (|z'|/|\nu^2 - z'^2|^{1/2})|V(k, z')|\,|\mathrm{d}z'|\right] \quad \text{(A.8)}$$

for z on straight line part of $\Gamma(\nu)$, and for $z \in S$

$$|u_{\nu-\frac{1}{2}}(z)| < c_1(|z|^{1/2}/|\nu^2 - z^2|^{1/4})(1 + 2B_1)\exp\left[2c_1^2 \int_\nu^z (|z'|/|\nu^2 - z'^2|^{1/2})|V(k, z')|\right] \quad \text{(A.9)}$$

where B_1 is an upper bound to the contribution to $B(\nu, k)$ of the integral in equation (A.5) from 0 to ν.

Using the bounds (A.7) to (A.9) in (A.6) we can prove in the same way that, for z on 0 to ν,

$$|u_{\nu-\frac{1}{2}}(k+\delta, z) - u_{\nu-\frac{1}{2}}(k, z)|$$
$$\leqslant 2c_1^2 \exp\{2c_1^2[I_1(\nu, k) + I_1(\nu, k+\delta)]\}I_2(\nu, k, \delta)$$
$$\times |z|^{1/2}/|\nu^2 - z^2|^{1/4})\exp[-\mathrm{Re}\,\nu(\alpha - \tanh\alpha)] \quad \text{(A.10)}$$

and for $z \in S$,

$$|u_{\nu-\frac{1}{2}}(k+\delta, z) - u_{\nu-\frac{1}{2}}(k, z)|$$
$$\leqslant (2|z|^{1/2}c_1^3/|\nu^2 - z^2|^{1/4})\exp\{2c_1^2[I_1(\nu, k) + I_1(\nu, k+\delta)]\}$$
$$\times I_2(\nu, k, \delta)\{1 + 4I_1(\nu, k+\delta)\exp[2c_1^2 I_1(\nu, k+\delta)]\} \quad \text{(A.11)}$$

where

$$I_1(\nu, k) = \int_{\Gamma(\nu)} (|z'|/|\nu^2 - z'^2|^{1/2})|V(k, z')|\,|dz'| \tag{A.12}$$

and

$$I_2(\nu, k, \delta) = \int_{\Gamma(\nu)} (|z'|/|\nu^2 - z'^2|^{1/2})|V(k+\delta, z') - V(k, z')|\,|dz'|. \tag{A.13}$$

The above bounds on $|u_{\nu-\frac{1}{2}}(k+\delta, z) - u_{\nu-\frac{1}{2}}(k, z)|$ may be used to derive a continuity condition on $A(\nu, k)$ since

$$\begin{aligned}
&|A(\nu, k+\delta) - A(\nu, k)| \\
&\quad \leqslant \left| \int_0^\infty (\pi z/2)^{1/2} J_\nu(z) V(k+\delta, z)[u_{\nu-\frac{1}{2}}(k+\delta, z) - u_{\nu-\frac{1}{2}}(k, z)]\,dz \right| \\
&\quad\quad + \left| \int_0^\infty (\pi z/2)^{1/2} J_\nu(z)[V(k+\delta, z) - V(k, z)] u_{\nu-\frac{1}{2}}(k, z)\,dz \right| \\
&\quad \leqslant c_1 I_2(\nu, k, \delta) \exp[2c^2 I_1(\nu, k)]\{1 + 2B_1 \\
&\quad\quad + c_1^2 \exp[2c_1^2 I_1(\nu, k+\delta)][I_1(\nu, k+\delta)]^2[2 + 4\exp(2c_1^2 I_1(\nu, k+\delta))]\}.
\end{aligned} \tag{A.14}$$

A similar bound holds for $B(\nu, k)$.

Bounds on the integrals $I_1(\nu, k)$ have been given by Bessis (1964) and are for all ν on C, and for $k > 0$

$$I_1(\nu, k) < c_2/|\nu|^{\gamma-1} \tag{A.15}$$

where c_2 is independent of ν, k. Similarly since the bounds (A.2) on $|V(k+\delta, z) - V(k, z)|$ are just $(\delta/k)(1 + 1/k^2)$ times those on $V(k, z)$ it follows that

$$I_2(\nu, k, \delta) < (c_3\delta/k)(1 + 1/k^2)/|\nu|^{\gamma-1} \tag{A.16}$$

where c_3 is independent of k, ν.

Finally we may write,

$$f(\nu, k+\delta) - f(\nu, k) = \frac{\begin{array}{r} A(\nu, k) - A(\nu, k+\delta) + i\{[A(\nu, k) - A(\nu, k+\delta)]B(\nu, k+\delta) \\ + A(\nu, k+\delta)[B(\nu, k+\delta) - B(\nu, k)]\} \end{array}}{[1 + iB(\nu, k)][1 + iB(\nu, k+\delta)]}. \tag{A.17}$$

Since $f(\nu, k+\delta), f(\nu, k)$ have no poles on C, i.e. $1 + iB(\nu, k)$, $1 + iB(\nu, k+\delta)$ have no zeroes on C, then for $\gamma > 1$ there exist $\epsilon_1 > 0$ such that

$$|1 + iB(\nu, k+\delta)|,\ |1 + iB(\nu, k)| > \epsilon_1 > 0 \tag{A.18}$$

for all ν on C.

Using the bounds (A.18), (A.16) and (A.14) in (A.17) one finally arrives at the continuity condition (2.2) on $f(\nu, k)$ for $\gamma \geqslant 1$.

Appendix B

In this Appendix we give the proof of theorem 2.3. We can write

$$\rho((k+\delta)^2,t)-\rho(k^2,t)$$
$$=[-\mathrm{i}/(k+\delta)]\int_C\{f(\nu,k+\delta)f^*(\nu^*,k+\delta)-f(\nu,k)f^*(\nu^*,k)\}$$
$$\times P_{\nu-\frac{1}{2}}[1+t/2(k+\delta)^2]\nu\,\mathrm{d}\nu\,[-\mathrm{i}/(k+\delta)]\int_C f(\nu,k)f^*(\nu^*,k)$$
$$\times\{P_{\nu-\frac{1}{2}}[1+t/2(k+\delta)^2]-P_{\nu-\frac{1}{2}}(1+t/2k^2)\}\nu\,\mathrm{d}\nu$$
$$\times[-\mathrm{i}\delta/k(k+\delta)]\int_C f(\nu,k)f^*(\nu^*,k)P_{\nu-\frac{1}{2}}(1+t/2k^2)\nu\,\mathrm{d}\nu. \qquad \text{(B.1)}$$

Let I_i $(i=1,2,3)$ be the three contour integrals on the RHS of (B.1). We further write $I_i=J_{i,1}+J_{i,2}$, where $J_{i,1}$, $J_{i,2}$ are the contributions to I_i from the straight-line part Γ_1 and curved part Γ_2 of C respectively.

On Γ_1, the straight line part of C, let $\nu=\frac{1}{4}+\mathrm{i}y$ with y real. Use is made of bounds on Legendre functions and their derivatives for these values of ν given by Atkinson and Frederiksen (1975). In particular, using inequality (A.22) of that work,

$$|P_{\nu-\frac{1}{2}}[1+t/2(k+\delta)^2]|\leqslant\eta_1(k)|\nu|^{-1/2}t^{-1/4} \qquad t>0,\ \nu=\tfrac{1}{4}+\mathrm{i}y \qquad \text{(B.2)}$$

where $\eta_1(k)$ is a known function.

Now

$$|J_{1,1}|=|1/(k+\delta)|\left|\int_{\Gamma_1}[f(\nu,k+\delta)f^*(\nu^*,k+\delta)-f(\nu,k)f^*(\nu^*,k)]\right.$$
$$\left.\times P_{\nu-\frac{1}{2}}[1+t/2(k+\delta)^2]\nu\,\mathrm{d}\nu\right|$$
$$\leqslant|1/k|\int_{\Gamma_1}|f(\nu,k+\delta)[f^*(\nu^*,k+\delta)-f^*(\nu^*,k)]$$
$$+f^*(\nu^*,k)[f(\nu,k+\delta)-f(\nu,k)]|^{\epsilon}$$
$$\times|\lambda_1^2/\nu^{2\gamma-1}|^{1-\epsilon_1}|P_{\nu-\frac{1}{2}}[1+t/2(k+\delta)^2]|\,|\nu\,\mathrm{d}\nu| \qquad \text{(B.3)}$$

for any $0<\epsilon_1<1$, where the bound (2.1) has been used. Then using (B.2), (2.1) (2.2),

$$|J_{1,1}|\leqslant\left|\frac{\eta_2(k)}{kt^{1/4}}\int_{\Gamma_1}\frac{\delta^{\epsilon_1}\,\mathrm{d}|\nu|}{|\nu|^{(2\gamma-\frac{3}{2})\epsilon_1+(1-\epsilon_1)(2\gamma-1)-\frac{1}{2}}}\right| \qquad \text{(B.4)}$$

where $\eta_2(k)$ is independent of t. When $\gamma>\frac{5}{4}$ one can choose ϵ_1 sufficiently close to 0 to make integral convergent so that

$$|J_{1,1}|\leqslant\eta_3(k)\delta^{\epsilon_1}/t^{1/4} \qquad \text{(B.5)}$$

where ϵ_1 depends on γ.

Similarly using equation (A.23) of Atkinson and Frederiksen (1975) one can prove the bound

$$|P_{\nu-\frac{1}{2}}[1+t/2(k+\delta)^2]-P_{\nu-\frac{1}{2}}[1+t/2k^2]|\leqslant\eta_4(k)|\nu|^{1/2}t^{-1/4} \qquad \text{(B.6)}$$

for ν on Γ_1. Using this result

$$|J_{2,1}|=\left|[1/(k+\delta)]\int_{C_1}[f(\nu,k)f^*(\nu^*,k)][P_{\nu-\frac{1}{2}}[1+t/2(k+\delta)^2]\right.$$
$$\left.-P_{\nu-\frac{1}{2}}[1+t/2k^2]\}\nu\,\mathrm{d}\nu\leq\eta_4(k)\delta^{\epsilon_2}/t^{1/4}\right. \tag{B.7}$$

for some $\epsilon_2>0$ which depends on γ and η_4 is a known function of k. Finally using again (B.2), (2.1) when $\gamma>\frac{5}{4}$,

$$|J_{3,1}|=\left|[\delta/k(k+\delta)]\int_{C_1}f(\nu,k)f^*(\nu^*,k)P_{\nu-\frac{1}{2}}[1+t/2k^2]\nu\,\mathrm{d}\nu\right|\leq\eta_5(k)\delta/t^{1/4} \tag{B.8}$$

where $\eta_5(k)$ is independent of δ, t.

Let us look now at the contributions from the curved part Γ_2 of C. We use the bounds

$$|P_{\nu-\frac{1}{2}}(\cosh\alpha)|\leq\pi^{1/2}\exp[\alpha\,\mathrm{Re}(\nu-\tfrac{1}{2})] \tag{B.9}$$

and

$$|(dP_{\nu-\frac{1}{2}}/d\cosh\alpha)(\cosh\alpha)|\leq|\nu-\tfrac{1}{2}|\pi^{1/2}(1+|\coth\alpha|)\exp[\alpha\,\mathrm{Re}(\nu-\tfrac{3}{2})]. \tag{B.10}$$

From the latter inequality with $\cosh\alpha=1+t/2k^2$,

$$|P_{\nu-\frac{1}{2}}[1+t/2(k+\delta)^2]-P_{\nu-\frac{1}{2}}[1+t/2k^2]|$$
$$\leq(\delta t\pi^{1/2}/k^3)|\nu-\tfrac{1}{2}|[1+|\coth\alpha|]\times\exp[\alpha\,\mathrm{Re}(\nu-\tfrac{3}{2})] \tag{B.11}$$

Now on Γ_2, $\mathrm{Re}(\nu-\frac{1}{2})\leq L_0-\sigma-\frac{1}{2}$, so that on γ_2

$$|P_{\nu-\frac{1}{2}}(\cosh\alpha)|\leq\pi^{1/2}(2+t/k^2)^{L_0-\sigma-\frac{1}{2}} \tag{B.12}$$

and

$$|P_{\nu-\frac{1}{2}}[1+t/2(k+\delta)^2]-P_{\nu-\frac{1}{2}}[1+t/2k^2]|$$
$$\leq\frac{\delta t\pi^{1/2}}{k^3}|\nu|(2+t/k^2)^{L_0-\frac{3}{2}-\sigma}\times\left\{1+\frac{1+t/2k^2}{[(1+t/2k^2)^2-1]^{1/2}}\right\}. \tag{B.13}$$

Therefore, since Γ_2 is of finite length, the bounds (2.1), (2.2), (B.12), (B.13) may be used to put bounds on the contributions to the RHS of (B.1) coming from the integrals over Γ_2. They are

$$|J_{1,2}|\leq\eta_6(k)\delta t^{L_0-\sigma-\frac{1}{2}} \tag{B.14a}$$
$$|J_{3,2}|\leq\eta_7(k)\delta t^{L_0-\sigma-\frac{1}{2}} \tag{B.14b}$$
$$|J_{2,2}|\leq\eta_8(k)\delta^{\epsilon_3}t^{L_0-\sigma-\frac{1}{2}} \tag{B.14c}$$

with the η_i depending on k. We can combine equations (B.5), (B.7), (B.8) and (B.14) to give the bound (2.5) on $|\rho[(k+\delta)^2,t]-\rho(k^2,t)|$, remembering that the contour C is chosen so that $L_0-\sigma>\frac{1}{2}$.

Appendix C

When $V(r)$ has the representation (2.30),

$$f_l^B(k)=(1/2k)\int_\mu^\infty\phi(u)Q_l(1+u^2/2k^2)\,\mathrm{d}u \tag{C.1}$$

By standard methods as in (Atkinson and Frederiksen 1972), one can use this form for $f_l^{\mathrm{B}}(k)$ to prove that

$$\sum_{l=0}^{\infty}(2l+1)|f_l^{\mathrm{B}}(k)|^2 P_l(z)$$

$$=(1/8k^2)\int_{\mu}^{\infty}\int_{\mu}^{\infty}\phi(u_1)\phi(u_2)\int_{z_+(z_1,z_2)}^{\infty}[\mathrm{d}z'/(z'-z)]h^{-1/2}(z',z_1,z_2) \qquad \text{(C.2)}$$

where $z_\pm(z_1, z_2)=z_1z_2\pm(z_1^2-1)^{1/2}(z_2^2-1)^{1/2}$ with $z_i=1+u_i^2/2k^2$ and

$$h(z',z_1,z_2)=z'^2+z_1^2+z_2^2-2z'z_1z_2-1.$$

Interchanging the orders of integration in (C.2) we find that

$$\sum_{l=0}^{\infty}(2l+1)|f_l^{\mathrm{B}}(k)|^2P_l(1+t/2k^2)-\sum_{l=0}^{\infty}(2l+1)|f_l^{\mathrm{B}}(k)|^2$$

$$=(t/8k^2)\int_{(4\mu^2+\mu^2/k^2)}^{\infty}\bigg\{\int_{\mu}^{z_-[(1+t'/2k^2),(1+\mu^2/2k^2)]}\phi(u_2)\,\mathrm{d}u_2$$

$$\times\int_{\mu}^{z_-[(1+t'/2k^2),(1+u^2/2k^2)]}\phi(u_1)h^{-1/2}[(1+t/2h^2),(1+u_1^2/2h^2),$$

$$\times(1+u_2^2/2h^2)]\,\mathrm{d}u_1\bigg\}\mathrm{d}t'/t'(t'-t). \qquad \text{(C.3)}$$

It can be proved that with the prescribed limiting behaviour of $\phi(u)$ the triple integral is absolutely integrable for t not on the cut from $4\mu^2+\mu^2/k^2$ to infinity, so that this interchange of order of integration is allowed. Therefore we may write

$$\sum_{l=0}^{\infty}(2l+1)|f_l^{\mathrm{B}}(k)|^2P_l(1+t/2k^2)$$

$$=\sum_{l=0}^{\infty}(2l+1)|f_l^{\mathrm{B}}(k)|^2+(t/\pi)\int_{\mu^2+\mu^2/k^2}^{\infty}\rho_{\mathrm{B}}(s,t')/t'(t'-t) \qquad \text{(C.4)}$$

where

$$\rho_{\mathrm{B}}(s,t)=(1/8k^2)\int_{\mu}^{z_-[(1+t/2k^2),(1+\mu^2/2k^2)]}\phi(u_2)\,\mathrm{d}u_2$$

$$\times\int_{\mu}^{z_-[(1+t/2k^2),(1+u_2^2/2k^2)]}\phi(u_1)h^{-1/2}[(1+t/2k^2),$$

$$\times(1+u_1^2/2k^2),(1+u_2^2/2k^2)]\,\mathrm{d}u_1. \qquad \text{(C.5)}$$

Using the given bounds on the behaviour of $\phi(u)$ as $u\to\infty$ and $u\to\mu$, it is straightforward but tedious to show that

$$|\rho_{\mathrm{B}}(s,t)|\leqslant\eta_9\theta(t-4\mu^2-\mu^4/k^2)[k^{\frac{3}{2}-2\gamma}t^{\gamma/2-\frac{3}{4}}(2+t/2k^2)^{2-\frac{3}{2}\gamma}(1+\mu^2/k^2)^{\frac{5}{2}-2\gamma}$$

$$\times(t-4\mu^2-\mu^4/k^2)^{-\frac{5}{2}+2\gamma}+t^{\rho/2-\frac{1}{2}}/k+t^{-\frac{3}{2}+\gamma}/k+k^{-1/2}(4k^2+\mu^2)^{-1/4}$$

$$\times t^{\frac{1}{2}-\gamma/2}(2+t/2k^2)^{\frac{3}{4}-\gamma/2}] \qquad \text{(C.6)}$$

where η_9 is independent of t, k.

We can also derive a continuity condition for $\rho_{\mathrm{B}}(s,t)$ analogous to (2.5). The method is to use continuity conditions on the Mandelstam kernel $h^{-1/2}$ given by Kupsch

(1970). They are that defining

$$K(s, t, t_1, t_2) = \{8(k^2+\mu^2)^{1/2}k^3\}^{-1}h^{-1/2}(1+(t/2s), \\ \times 1+(t_1/2s), 1+(t_2/2s)) \qquad \text{if } (s, t, t_1, t_2) \in \mathrm{D} \\ = 0 \qquad \text{if } (s, t, t_1, t_2) \notin \mathrm{D} \tag{C.7}$$

then if $s' > s$, $0 < \delta_1 < \frac{1}{2}$

$$0 \leqslant K(s, t, t_1, t_2) - K(s', t, t_1, t_2) \leqslant |K(s, t, t_1, t_2)|^{1+2\delta_1}|2t^2s^2(s'-s)/s'|^{\delta_1} \tag{C.8}$$

when $(s, t, t_1, t_2) \in \mathrm{D}$, and

$$|K(s, t, t_1, t_2) - K(s, t', t_1, t_2)| \leqslant |K(s', t, t_1, t_2)|^{1+2\delta_1}|t^2s'(s'-s)^{\delta_1}| \tag{C.9}$$

when $(s, t, t_1, t_2) \notin \mathrm{D}$ and $(s', t, t_1, t_2) \in \mathrm{D}$. Here D is the domain of values of (s, t, t_1, t_2) such that

$$(1+2t/s)^2 + (1+2t_1/s)^2 + (1+2t_2/s)^2 - 2(1+2t/s)(1+2t_1/s)(1+2t_2/s) - 1 \geqslant 0.$$

Combining these inequalities with the bounds on $\phi(u)$, one finds after some labour that

$$|\rho_{\mathrm{B}}(s, t) - \rho_{\mathrm{B}}(s_1, t)| \\ \leqslant \eta_{10}(s, s_1)\{[|s-s_1|/(k^2+\mu^2)]|\rho_{\mathrm{B}}(s, t)| \\ + (|s-s_1|^{\delta_1}/|k^2+\mu^2|^{1/2})t^{2\delta_1}[t^{\frac{5}{4}+3\delta_1/2-\gamma}(t-4\mu^2-\mu^4/k^4)^{-\frac{5}{2}+2\gamma-\delta_1} \\ + t^{\rho/2-\frac{1}{2}+\delta_1} + t^{\gamma-\frac{3}{4}-\delta_1/2} + t^{\frac{5}{4}+\gamma+3\delta_1/2}]\} \\ \times \theta(t-4\mu^2-\mu^4/k^2) + (s_1 \Leftrightarrow s) \tag{C.10}$$

where η_{10} is independent of t and $0 < \delta_1 < \frac{1}{2}$.

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